

Asymptotics and statistical inferences on independent and non-identically distributed bivariate Gaussian triangular arrays

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Abstract In this paper, we establish the first and the second-order asymptotics of distributions of normalized maxima of independent and non-identically distributed bivariate Gaussian triangular arrays, where each vector of the n th row follows from a bivariate Gaussian distribution with correlation coefficient being a monotone continuous positive function of i/n . Furthermore, parametric inference for this unknown function is studied. Some simulation study and real data sets analysis are also presented.

Key words Bivariate Gaussian random vector; Maximum; Limiting distribution; Second-order expansion; Estimation.

AMS 2000 subject classification Primary 62E20, 60G70; Secondary 60F15, 60F05.

Running Title Asymptotics and statistical inferences on bivariate Gaussian triangular arrays

1 Introduction

Let $\{(X_{ni}, Y_{ni}), 1 \leq i \leq n, n \geq 1\}$ be independent bivariate Gaussian triangular arrays, and let ρ_{ni} denote the correlation coefficient of (X_{ni}, Y_{ni}) , $1 \leq i \leq n$. The bivariate maxima \mathbf{M}_n are defined componentwise by

$$\mathbf{M}_n = (M_{n1}, M_{n2}) = \left(\max_{1 \leq i \leq n} X_{ni}, \max_{1 \leq i \leq n} Y_{ni} \right).$$

For the asymptotic distribution of \mathbf{M}_n , Sibuya (1960) showed that M_{n1} and M_{n2} are asymptotic independent if $\rho_{ni} = \rho \in (-1, 1)$, which coincides with the tail asymptotic independence of Gaussian copula, see Embrechts et al. (2002). For the case of $\rho_{ni} = \rho_n$, Hüsler and Reiss (1989) derived that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}, y \in \mathbb{R}} \left| \mathbb{P} \left(M_{n1} \leq b_n + \frac{x}{b_n}, M_{n2} \leq b_n + \frac{y}{b_n} \right) - H_\lambda(x, y) \right| = 0 \quad (1.1)$$

provided that the following Hüsler-Reiss condition

$$\lim_{n \rightarrow \infty} b_n^2 (1 - \rho_n) = 2\lambda^2 \quad (1.2)$$

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holds with $\lambda \in [0, \infty]$ (the converse assertion is proved by Kabluchko et al. (2009)), where the norming constant b_n satisfies

$$1 - \Phi(b_n) = \frac{1}{n}, \quad (1.3)$$

where $H_\lambda(x, y)$, the so-called Hüsler-Reiss max-stable distribution, is given by

$$H_\lambda(x, y) = \exp \left(-\Phi \left(\lambda + \frac{x-y}{2\lambda} \right) e^{-y} - \Phi \left(\lambda + \frac{y-x}{2\lambda} \right) e^{-x} \right)$$

with $\Phi(x)$ standing for the standard Gaussian distribution. Obviously, components of \mathbf{M}_n are asymptotic dependent when $\lambda < \infty$.

For Hüsler-Reiss model, the asymptotic behavior of the dynamic copula version of normalized \mathbf{M}_n has also been studied in recent literature. Under the Hüsler-Reiss condition (1.2), Frick and Reiss (2013) considered the asymptotic behaviors of the distribution of $(n(\max_{1 \leq i \leq n} \Phi(X_{ni}) - 1), n(\max_{1 \leq i \leq n} \Phi(Y_{ni}) - 1))$. Allowing ρ_{ni} to depend on both i and n , Liao et al. (2014a) extended the result in Frick and Reiss (2013) by assuming that

$$\rho_{ni} = 1 - m(i/n)/\log n \quad (1.4)$$

for some positive function $m(x)$. For other work related to Hüsler-Reiss model and its extensions, see, e.g., Hashorva (2005, 2006), Hashorva et al. (2012), Hashorva and Weng (2013), Kabluchko (2011), Engelke et al. (2014), Das et al. (2014) and reference therein.

The objective of this paper is to derive the first and the second-order distributional expansions of the dynamic Hüsler-Reiss model with ρ_{ni} given by (1.4) and establish statistical inferences related to the function $m(x)$. For the convergence rates and higher-order expansions of univariate extremes, we refer to de Haan and Resnick (1996), Nair (1981), Liao et al. (2014b) and reference therein. For the convergence rates of bivariate extremes, see de Haan and Peng (1997) for the general case. For the special case of the bivariate Hüsler-Reiss model, Hashorva et al. (2014) established the higher-order distributional expansions of \mathbf{M}_n , and Liao and Peng (2014) established the uniform convergence rate of (1.1). Liao and Peng (2015) also derived the second-order expansion of the joint distribution of normalized maximum and minimum. So far, there are no studies on the convergence and distributional expansion of \mathbf{M}_n under the assumption that (X_{ni}, Y_{ni}) 's are not identically distributed. The main goal of this paper is to fill this gap. Borrowing the ideas from Liao et al. (2014a), we derive in this paper the limit distribution of the normalized maxima \mathbf{M}_n if the function $m(i/n)$ in (1.4) satisfies some regular conditions, and establish its second-order distributional expansion provided that the convergence rate of $\max_{1 \leq i \leq n} m(i/n)$ is given. Furthermore, parametric estimation of $m(x)$ is considered through maximum likelihood estimation. The asymptotic properties of the estimators can be employed to test the condition proposed by Hüsler and Reiss (1989).

The rest of this paper is organized as follows. In section 2, we provide the main results and statistical procedures. A simulation study and some real data analysis are presented in Section 3. All proofs are given in Section 4.

2 Methodology

2.1 Convergence of maxima

In this section, the limiting distribution and the second-order expansion of distribution of normalized \mathbf{M}_n are provided with ρ_{ni} satisfying (1.4). The first result is about the first-order asymptotic which is stated as follows.

Theorem 1. *Under the condition (1.4),*

(i) *if $\max_{1 \leq i \leq n} m(i/n) \rightarrow 0$, then for any $x, y \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) = \Lambda(\min(x, y));$$

(ii) *if $\min_{1 \leq i \leq n} m(i/n) \rightarrow \infty$, then for any $x, y \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) = \Lambda(x)\Lambda(y);$$

(iii) *if $m(s)$ is a continuous positive function on $[0, 1]$, then for any $x, y \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) = H(x, y)$$

with

$$H(x, y) = \exp \left(-e^{-y} \int_0^1 \Phi \left(\sqrt{m(t)} + \frac{x-y}{2\sqrt{m(t)}} \right) dt - e^{-x} \int_0^1 \Phi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt \right).$$

To establish the second-order distributional expansion of normalized maxima, we consider the following three cases in turn: $m(t)$ is monotone and continuous on $[0, 1]$; $\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} m(i/n) = 0$; and $\lim_{n \rightarrow \infty} \min_{1 \leq i \leq n} m(i/n) = \infty$.

Theorem 2. *Under the condition (1.4), assume that $m(t)$ is monotone and continuous on $[0, 1]$, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\log n}{\log \log n} \left(\mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) - H(x, y) \right) \\ &= \frac{1}{2} \left(\int_0^1 \sqrt{m(t)} \varphi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt \right) e^{-x} H(x, y), \end{aligned} \quad (2.1)$$

where $\varphi(x)$ is the probability density function of standard Gaussian distribution.

Theorem 3. *Let the norming constant b_n be given by (1.3). Assume that $\lim_{n \rightarrow \infty} (\log n)^4 \max_{1 \leq i \leq n} m(i/n) = 0$, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\log n) \left(\mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) - \Lambda(\min(x, y)) \right) \\ &= \frac{1}{4} (\min(x, y))^2 + 2 \min(x, y) e^{-\min(x, y)} \Lambda(\min(x, y)). \end{aligned} \quad (2.2)$$

Theorem 4. *Let the norming constant b_n be given by (1.3). Assume that $\lim_{n \rightarrow \infty} (\log \log n) / (\min_{1 \leq i \leq n} m(i/n)) = 0$, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\log n) \left(\mathbb{P}(M_{n1} \leq b_n + x/b_n, M_{n2} \leq b_n + y/b_n) - \Lambda(x)\Lambda(y) \right) \\ &= \left(\frac{x^2 + 2x}{4} e^{-x} + \frac{y^2 + 2y}{4} e^{-y} \right) \Lambda(x)\Lambda(y). \end{aligned} \quad (2.3)$$

2.2 Parametric inference

Now we consider statistical inference for fitting a parametric form to the unknown function $m(s)$. Here we consider the family $m(s) = \alpha + \beta s^\gamma$, where $\alpha > 0$, $\beta \neq 0$, $\gamma > 0$. Note that when $\beta = 0$, γ can not be identified, and when $\gamma = 0$, α and β can't be distinguished, cf. Liao et al. (2014a).

We use the maximum likelihood estimation (MLE) to get the estimator, which is

$$(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \arg \max_{(\alpha, \beta, \gamma)} \left(-n \log 2\pi - \frac{1}{2} \sum_{i=1}^n \log(1 - \rho_{ni}^2) - \sum_{i=1}^n \frac{X_{ni}^2 + Y_{ni}^2}{2(1 - \rho_{ni}^2)} + \sum_{i=1}^n \frac{\rho_{ni}}{1 - \rho_{ni}^2} X_{ni} Y_{ni} \right).$$

That is, $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is the solution to the following score equations

$$\begin{cases} l_{n1}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\frac{\rho_{ni}}{(\log n)(1 - \rho_{ni}^2)} + \frac{(1 + \rho_{ni}^2)X_{ni}Y_{ni}}{(\log n)(1 - \rho_{ni}^2)^2} - \frac{\rho_{ni}(X_{ni}^2 + Y_{ni}^2)}{(\log n)(1 - \rho_{ni}^2)^2} \right) = 0, \\ l_{n2}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\frac{\rho_{ni}}{(\log n)(1 - \rho_{ni}^2)} + \frac{(1 + \rho_{ni}^2)X_{ni}Y_{ni}}{(\log n)(1 - \rho_{ni}^2)^2} - \frac{\rho_{ni}(X_{ni}^2 + Y_{ni}^2)}{(\log n)(1 - \rho_{ni}^2)^2} \right) \left(\frac{i}{n} \right)^\gamma = 0, \\ l_{n3}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\frac{\rho_{ni}}{(\log n)(1 - \rho_{ni}^2)} + \frac{(1 + \rho_{ni}^2)X_{ni}Y_{ni}}{(\log n)(1 - \rho_{ni}^2)^2} - \frac{\rho_{ni}(X_{ni}^2 + Y_{ni}^2)}{(\log n)(1 - \rho_{ni}^2)^2} \right) \left(\frac{i}{n} \right)^\gamma \log\left(\frac{i}{n}\right) = 0. \end{cases} \quad (2.4)$$

The following theorem gives the asymptotic normality of the proposed estimator.

Theorem 5. Assume that (1.4) holds with $m(s) = \alpha + \beta s^\gamma$ for some $\alpha > 0$, $\beta \neq 0$, $\gamma > 0$. Then we have

$$\hat{\Delta} \left(\sqrt{n}(\hat{\alpha} - \alpha), \sqrt{n}(\hat{\beta} - \beta), \sqrt{n}(\hat{\gamma} - \gamma) \right)^T \xrightarrow{d} N(0, \Sigma), \quad (2.5)$$

where the matrices $\hat{\Delta}$ and Σ are given by

$$\hat{\Delta} = \begin{pmatrix} \int_0^1 \frac{1}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{t^{\hat{\gamma}}}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{\hat{\beta}t^{\hat{\gamma}} \log t}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt \\ \int_0^1 \frac{t^{\hat{\gamma}}}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{t^{2\hat{\gamma}}}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{\hat{\beta}t^{2\hat{\gamma}} \log t}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt \\ \int_0^1 \frac{t^{\hat{\gamma}} \log t}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{t^{2\hat{\gamma}} \log t}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt & \int_0^1 \frac{\hat{\beta}t^{2\hat{\gamma}} (\log t)^2}{2(\hat{\alpha} + \hat{\beta}t^{\hat{\gamma}})^2} dt \end{pmatrix}.$$

and

$$\Sigma = \begin{pmatrix} \int_0^1 \frac{1}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^\gamma}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^\gamma \log t}{2(\alpha + \beta t^\gamma)^2} dt \\ \int_0^1 \frac{t^\gamma}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma}}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma} \log t}{2(\alpha + \beta t^\gamma)^2} dt \\ \int_0^1 \frac{t^\gamma \log t}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma} \log t}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma} (\log t)^2}{2(\alpha + \beta t^\gamma)^2} dt \end{pmatrix} \quad (2.6)$$

Another interesting parametric form is $m(s) = \alpha + \beta s$ for some $\alpha > 0$, $\beta \in \mathbb{R}$. In this case, when $\beta = 0$, $m(s)$ becomes constant, which means that the observations $(X_1, Y_1), \dots, (X_n, Y_n)$ are independent and identically distributed random vectors.

Theorem 6. Suppose (1.4) holds with $m(s) = \alpha + \beta s$ for some $\alpha > 0$, $\beta \neq 0$. Then we have

$$\begin{pmatrix} \sqrt{n} \left(\frac{1}{2\hat{\beta}} \log \left(1 + \frac{\hat{\beta}}{\hat{\alpha}} \right) - \left(\frac{\hat{\beta}\alpha - \hat{\alpha}\beta}{2\hat{\alpha}\hat{\beta}(\hat{\alpha} + \hat{\beta})} + \frac{\beta}{2\hat{\beta}^2} \log \left(1 + \frac{\hat{\beta}}{\hat{\alpha}} \right) \right) \right) \\ \sqrt{n} \left(\frac{1}{2\hat{\beta}} - \frac{\hat{\alpha}}{2\hat{\beta}^2} \log \left(1 + \frac{\hat{\beta}}{\hat{\alpha}} \right) - \left(\frac{\beta}{2\hat{\beta}^2} + \frac{\hat{\alpha}\beta - \hat{\beta}\alpha}{2\hat{\beta}^2(\hat{\alpha} + \hat{\beta})} - \frac{2\hat{\alpha}\beta - \hat{\beta}\alpha}{2\hat{\beta}^3} \log \left(1 + \frac{\hat{\beta}}{\hat{\alpha}} \right) \right) \right) \end{pmatrix} \xrightarrow{d} N(0, \tilde{\Sigma}), \quad (2.7)$$

where $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma} = \begin{pmatrix} \frac{1}{2\alpha(\alpha + \beta)} & -\frac{1}{2\beta(\alpha + \beta)} + \frac{1}{2\beta^2} \log \left(1 + \frac{\beta}{\alpha} \right) \\ -\frac{1}{2\beta(\alpha + \beta)} + \frac{1}{2\beta^2} \log \left(1 + \frac{\beta}{\alpha} \right) & \frac{1}{2\beta^2} \left(1 + \frac{\alpha}{\alpha + \beta} - \frac{2\alpha}{\beta} \log \left(1 + \frac{\beta}{\alpha} \right) \right) \end{pmatrix}.$$

3 Simulation and data analysis

In this section we examine the finite sample performance of the proposed estimators by drawing independent $(X_{n1}, Y_{n1}), \dots, (X_{nn}, Y_{nn})$ with (X_{ni}, Y_{ni}) following the bivariate Gaussian distribution with coefficient $\rho_{ni} = 1 - m(i/n)/\log n$. We consider $n = 1000, 3000$ or 10000 , and repeat 1000 times.

First we consider $m(s) = \alpha$ with $\alpha = 1$ or 10 , and calculate the average and mean squared error for $\hat{\alpha}$. We can observe from Table 1 that i) the averages of $\hat{\alpha}$ is near by the true value α ; ii) small mean squared errors show the robustness of $\hat{\alpha}$. Next the case of $m(s) = \alpha + \beta s$ is considered. Table 2 reports the averages and mean squared errors for estimator $(\hat{\alpha}, \hat{\beta})$. As n becomes large, the accuracy of all estimators improve. Finally, we consider the case of $m(s) = \alpha + \beta s^\gamma$ with sample size $n = 10000$. The simulation shows that all estimators are closer to their true values with small mean squared errors, cf. Table 3 for details.

Table 1: Estimators for the case of $m(s) = \alpha$

	$\alpha = 1$	$\alpha = 10$	$\alpha = 1$	$\alpha = 10$	$\alpha = 1$	$\alpha = 10$
	$n = 1000$	$n = 1000$	$n = 3000$	$n = 3000$	$n = 10000$	$n = 10000$
$\mathbf{E}(\hat{\alpha})$	0.9980043	9.992257	1.001875	9.997267	1.00005	9.999436
$\text{MSE}(\hat{\alpha})$	0.002109944	0.02555685	0.0006846334	0.01812748	0.0002033414	0.007965965

Table 2: Estimators for the case of $m(s) = \alpha + \beta s$ with $\alpha = 1$.

	$\beta = 1$	$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$	$\beta = 0$
	$n = 1000$	$n = 1000$	$n = 3000$	$n = 3000$	$n = 10000$	$n = 10000$
$\mathbf{E}(\hat{\alpha})$	1.002458	1.002368	1.001585	0.9973226	0.9996816	1.000073
$\text{MSE}(\hat{\alpha})$	0.01125451	0.00864667	0.00385037	0.002605236	0.00124363	0.000797927
$\mathbf{E}(\hat{\beta})$	0.9978728	-0.000328	0.9967	0.004754433	1.001072	0.000712807
$\text{MSE}(\hat{\beta})$	0.04707124	0.02588997	0.01669121	0.007914633	0.00509091	0.002447169

Table 3: Estimators for the case of $m(s) = \alpha + \beta s^\gamma$ with $\alpha = \beta = 1$.

	$\mathbf{E}(\hat{\alpha})$	$\text{MSE}(\hat{\alpha})$	$\mathbf{E}(\hat{\beta})$	$\text{MSE}(\hat{\beta})$	$\mathbf{E}(\hat{\gamma})$	$\text{MSE}(\hat{\gamma})$
$\gamma = 0.5$	0.9437925	0.05247714	1.058977	0.04504459	0.5092629	0.03055645
$\gamma = 1$	0.994503	0.004061697	1.009859	0.005172654	1.01609	0.03928909
$\gamma = 1.5$	0.9955838	0.001943133	1.003917	0.005124053	1.504647	0.06171575

For the applications, we consider four couples of real data sets: The first is the log-returns of the exchange rates between US dollar and British pound and those between Canadian dollar and British pound from April 3, 2000 to November 11, 2014. The second is the log-returns of the Shanghai Stock Exchange composite

index (SSE Composite) and ShenZhen Stock Exchange Composite index (SZSE Composite) from March 4, 1996 to November 12, 2014. The third is the log-returns of the CSI 300 index and CSI 300 index futures from April 16, 2010 to November 13, 2014. The forth is the wave and surge heights in southwest England which comprise 2894 wave heights and 2894 surge heights. All time series are plotted in Figure 1.

First, we calculate the i th sample correlation for each couple of the mentioned data sets by using $\{(X_{n1}, Y_{n1}), \dots, (X_{ni}, Y_{ni})\}$. Figures 2-5 show respectively that each tends to constant ultimately. Now we estimate the correlation $\rho = 1 - m(i/n)/\log(n)$ by assuming that $m(s)$ is a constant, which also are illustrated by Figures 2-5, respectively. The constancy of $m(s)$ shows that observations are identically distributed.

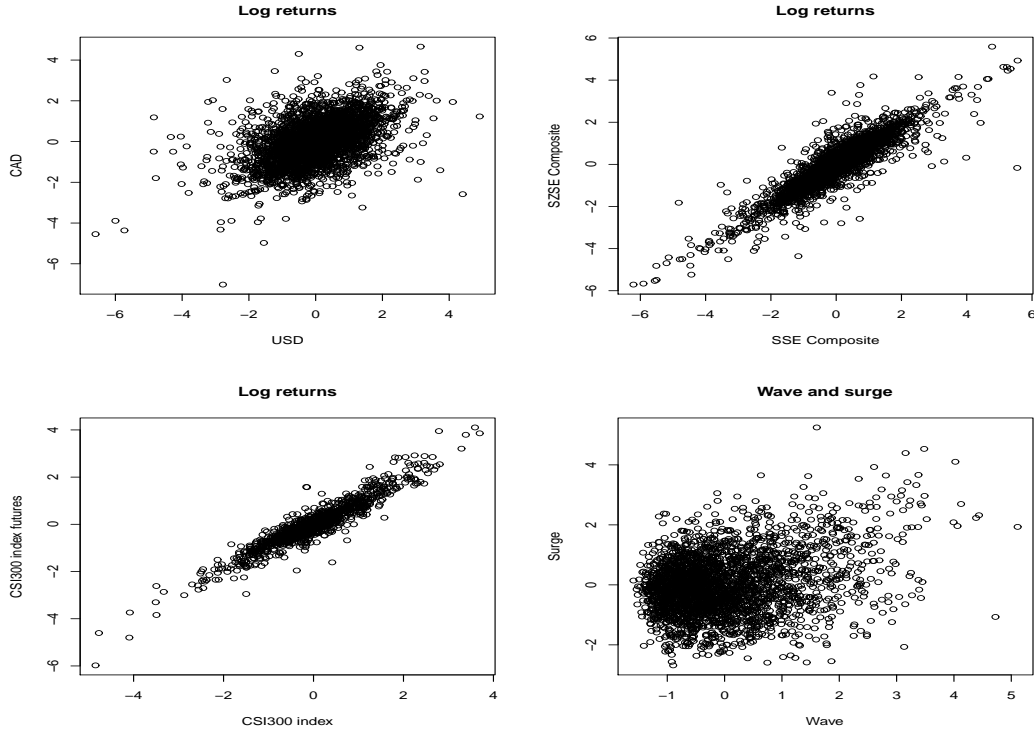


Figure 1: log-returns of the exchange rates between US dollar and British pound and those between Canadian dollar and British pound (top left); the log-returns of SSE Composite and SZSE Composite from (top right); the log-returns of the CSI 300 index and CSI 300 index futures (bottom left); the wave and surge heights in southwest England (bottom right).

4 Proofs

The aim of this section is to prove our main results. In the sequel, let $F_i(x, y)$ denote the distribution function of (X_{ni}, Y_{ni}) , $1 \leq i \leq n$; and let $u_n(x) = b_n + x/b_n$ for notational simplicity.

Proof of Theorem 1. We only consider the case (iii) here, since the other two cases can be derived by Slepian's Lemma and the result of case (iii).

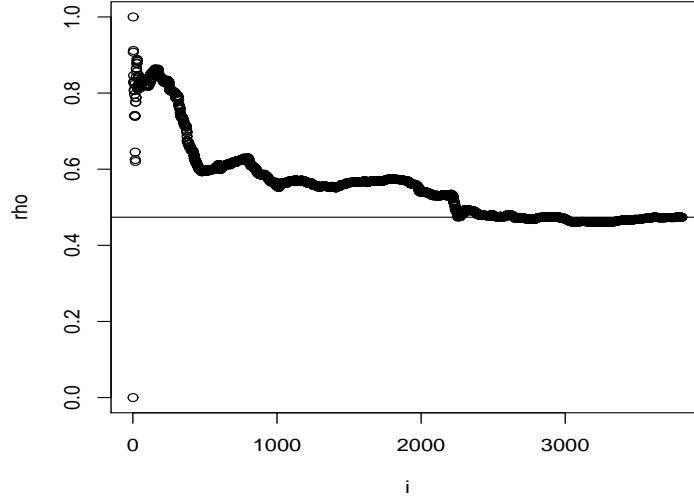


Figure 2: Exchange rates. Dotted line represents the sample correlations, and solid line represents the correlation estimate $\hat{\rho} = 0.4738478$ with $\hat{m} = 4.338189$.

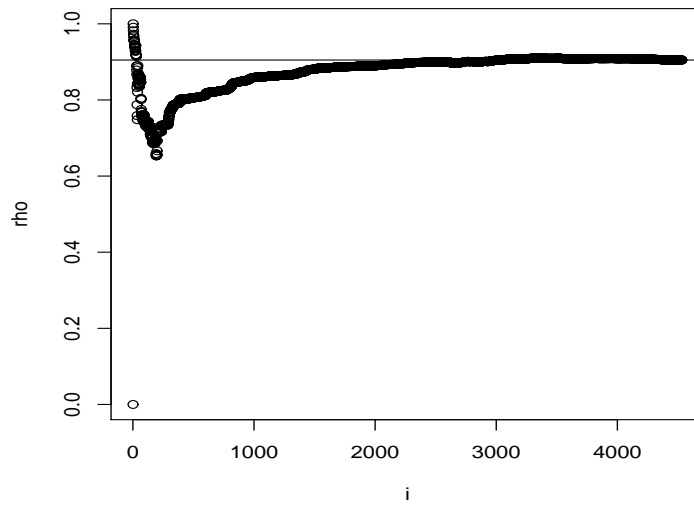


Figure 3: SSE Composite and SZSE Composite. Dotted line represents the sample correlations, and solid line represents the correlation estimate $\hat{\rho} = 0.9048648$ with $\hat{m} = 0.8009351$.

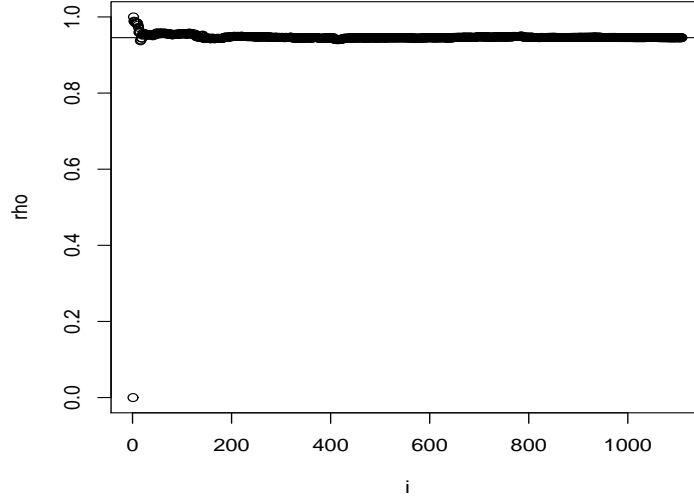


Figure 4: CSI 300 index and CSI 300 index futures. Dotted line represents the sample correlations, and solid line represents the correlation estimate $\hat{\rho} = 0.9455578$ with $\hat{m} = 0.3817058$.

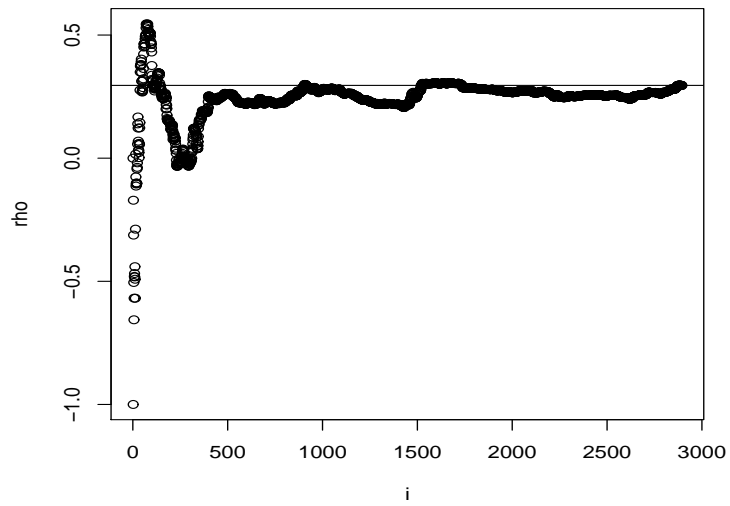


Figure 5: Wave and surge heights. Dotted line represents the sample correlations, and solid line represents the correlation estimate $\hat{\rho} = 0.2955482$ with $\hat{m} = 5.614759$.

It follows from (1.3) that

$$b_n = (2 \log n)^{\frac{1}{2}} - \frac{\log \log n + \log 4\pi}{2(2 \log n)^{\frac{1}{2}}} + o\left(\frac{1}{(\log n)^{\frac{1}{2}}}\right), \quad (4.1)$$

which implies that $b_n^2 \sim 2 \log n$ as $n \rightarrow \infty$. Combining with (1.4), we have

$$\begin{aligned} & \frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \\ &= \sqrt{\frac{1 - \rho_{ni}}{1 + \rho_{ni}}} b_n + \frac{x - z}{b_n \sqrt{1 - \rho_{ni}^2}} + \sqrt{\frac{1 - \rho_{ni}}{1 + \rho_{ni}}} \frac{z}{b_n} \\ &= \sqrt{\frac{m(\frac{i}{n})}{(\log n) \left(2 - \frac{m(\frac{i}{n})}{\log n}\right)}} (2 \log n)^{\frac{1}{2}} \left(1 - \frac{\log \log n + \log 4\pi}{4 \log n} + o\left(\frac{1}{\log n}\right)\right) \\ & \quad + \frac{x - z}{(2 \log n)^{\frac{1}{2}} \left(1 - \frac{\log \log n + \log 4\pi}{4 \log n} + o\left(\frac{1}{\log n}\right)\right)} \sqrt{\frac{m(\frac{i}{n})}{(\log n) \left(2 - \frac{m(\frac{i}{n})}{\log n}\right)}} \\ & \quad + \sqrt{\frac{m(\frac{i}{n})}{(\log n) \left(2 - \frac{m(\frac{i}{n})}{\log n}\right)}} \frac{z}{(2 \log n)^{\frac{1}{2}} \left(1 - \frac{\log \log n + \log 4\pi}{4 \log n} + o\left(\frac{1}{\log n}\right)\right)} \\ &= \sqrt{m\left(\frac{i}{n}\right)} + \frac{x - z}{2\sqrt{m\left(\frac{i}{n}\right)}} - \frac{\log \log n}{4 \log n} \sqrt{m\left(\frac{i}{n}\right)} + \frac{\log \log n}{8 \log n} \frac{x - z}{\sqrt{m\left(\frac{i}{n}\right)}} + O\left(\frac{1}{\log n}\right) + O\left(\frac{z}{\log n}\right) \end{aligned} \quad (4.2)$$

for large n .

By using the inequality $|\Phi(x) - \Phi(y)| \leq |x - y|$ for any $x, y \in \mathbb{R}$, we have

$$\left| \Phi\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}\right) - \Phi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x - z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) \right| \leq (1 + |z|)O\left(\frac{\log \log n}{\log n}\right) + (1 + |z|)O\left(\frac{1}{\log n}\right)$$

for large n and any $z \in \mathbb{R}$, which implies that

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}\right) \exp\left(-z - \frac{z^2}{2b_n^2}\right) dz \\ &= \frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x - z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) e^{-z} dz (1 + o(1)) \\ &\rightarrow \int_0^1 \int_y^\infty \Phi\left(\sqrt{m(t)} + \frac{x - z}{2\sqrt{m(t)}}\right) e^{-z} dz dt \\ &= -e^{-x} + e^{-y} \int_0^1 \Phi\left(\sqrt{m(t)} + \frac{x - y}{2\sqrt{m(t)}}\right) dt + e^{-x} \int_0^1 \Phi\left(\sqrt{m(t)} + \frac{y - x}{2\sqrt{m(t)}}\right) dt \end{aligned} \quad (4.3)$$

as $n \rightarrow \infty$.

Note that by Castro (1987) and (1.3),

$$n^{-1} = 1 - \Phi(b_n) = \frac{\varphi(b_n)}{b_n} (1 - b_n^{-2} + O(b_n^{-4})) \quad (4.4)$$

for large n , and Nair (1981) showed that

$$\lim_{n \rightarrow \infty} b_n^2 (-n(1 - \Phi(u_n(x))) + e^{-x}) = \frac{x^2 + 2x}{2} e^{-x}. \quad (4.5)$$

Combining with (4.3), we have

$$\begin{aligned}
& -\sum_{i=1}^n (1 - F_i(u_n(x), u_n(y))) \\
&= -n(1 - \Phi(u_n(x))) - n^{-1}(1 + b_n^{-2} + O(b_n^{-4})) \sum_{i=1}^n \int_y^\infty \Phi\left(\frac{u_n(x) - \rho_{ni}u_n(s)}{\sqrt{1 - \rho_{ni}^2}}\right) \exp\left(-s - \frac{s^2}{2b_n^2}\right) ds \\
&\rightarrow -e^{-x} - \int_0^1 \int_y^\infty \Phi\left(\sqrt{m(t)} + \frac{x-s}{2\sqrt{m(t)}}\right) e^{-s} ds dt \\
&= -e^{-y} \int_0^1 \Phi\left(\sqrt{m(t)} + \frac{x-y}{2\sqrt{m(t)}}\right) dt - e^{-x} \int_0^1 \Phi\left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}}\right) dt
\end{aligned} \tag{4.6}$$

as $n \rightarrow \infty$, which implies the desired result.

The proof is complete. \square

Proof of Theorem 2. By (4.2) we can get

$$\begin{aligned}
& -\frac{1}{n} \sum_{i=1}^n \int_y^\infty \varphi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) e^{-z} dz \\
&= \frac{\log \log n}{4 \log n} \frac{1}{n} \sum_{i=1}^n \int_y^\infty \varphi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) \left(\sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) e^{-z} dz + O\left(\frac{1}{\log n}\right) \\
&\sim \frac{\log \log n}{4 \log n} \int_0^1 \int_y^\infty \varphi\left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}}\right) \left(\sqrt{m(t)} - \frac{x-z}{2\sqrt{m(t)}}\right) e^{-z} dz dt \\
&= \frac{\log \log n}{2 \log n} e^{-x} \int_0^1 \sqrt{m(t)} \varphi\left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}}\right) dt
\end{aligned} \tag{4.7}$$

as $n \rightarrow \infty$.

By Taylor expansion with Lagrange reminder term, we have

$$\begin{aligned}
& \Phi\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}\right) \\
&= \Phi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) + \varphi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{2\sqrt{m\left(\frac{i}{n}\right)}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) \\
&\quad + \frac{1}{2} \theta \varphi(\theta) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right)^2,
\end{aligned}$$

where

$$\min\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}, \sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right) < \theta < \max\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}, \sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right).$$

Combining with (4.7) we have

$$-\frac{1}{n} \sum_{i=1}^n \int_y^\infty \left(\Phi\left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1 - \rho_{ni}^2}}\right) - \Phi\left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}}\right)\right) e^{-z} dz$$

$$\begin{aligned}
&= -\frac{1}{n} \sum_{i=1}^n \int_y^\infty \varphi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \\
&\quad - \frac{1}{2n} \sum_{i=1}^n \int_y^\infty \theta\varphi(\theta) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right)^2 e^{-z} dz \\
&\sim \frac{\log \log n}{2 \log n} e^{-x} \int_0^1 \sqrt{m(t)} \varphi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt
\end{aligned} \tag{4.8}$$

as $n \rightarrow \infty$ since

$$\frac{1}{n} \sum_{i=1}^n \int_y^\infty \theta\varphi(\theta) \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} - \sqrt{m\left(\frac{i}{n}\right)} - \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right)^2 e^{-z} dz = O \left(\left(\frac{\log \log n}{\log n} \right)^2 \right).$$

Now, we first assert that

$$\frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz - \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt = O \left(\frac{1}{n} \right) \tag{4.9}$$

holds for any $x, y \in \mathbb{R}$. Combining with (4.8), we can get

$$\begin{aligned}
&-\frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} \right) e^{-z} dz + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&\sim \frac{\log \log n}{2 \log n} e^{-x} \int_0^1 \sqrt{m(t)} \varphi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt
\end{aligned} \tag{4.10}$$

as $n \rightarrow \infty$.

From (4.4), (4.5) and (4.10), it follows that

$$\begin{aligned}
&-\sum_{i=1}^n (1 - F_i(u_n(x), u_n(y))) + e^{-x} + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= -n(1 - \Phi(u_n(x))) + e^{-x} - n^{-1} (1 + b_n^{-2} + O(b_n^{-4})) \sum_{i=1}^n \int_y^\infty \Phi \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz \\
&\quad + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= -n(1 - \Phi(u_n(x))) + e^{-x} - \frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} \right) e^{-z} dz + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&\quad - \frac{1}{nb_n^2} \sum_{i=1}^n \int_y^\infty \Phi \left(\frac{u_n(x) - \rho_{ni}u_n(z)}{\sqrt{1-\rho_{ni}^2}} \right) e^{-z} \left(1 - \frac{z^2}{2} \right) dz + O(b_n^{-4}) \\
&\sim \frac{\log \log n}{2 \log n} e^{-x} \int_0^1 \sqrt{m(t)} \varphi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt
\end{aligned}$$

as $n \rightarrow \infty$, which implies that

$$\mathbb{P}(M_{n1} \leq u_n(x), M_{n2} \leq u_n(y)) - H(x, y)$$

$$\begin{aligned}
&= H(x, y) \left(\exp \left(\sum_{i=1}^n \log F_i(u_n(x), u_n(y)) + e^{-x} + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \right) - 1 \right) \\
&= H(x, y) \left(- \sum_{i=1}^n (1 - F_i(u_n(x), u_n(y))) + e^{-x} + \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \right. \\
&\quad \left. - \frac{1}{2} \sum_{i=1}^n (1 - F_i(u_n(x), u_n(y)))^2 (1 + o(1)) \right) (1 + o(1)) \\
&\sim \frac{\log \log n}{2 \log n} \left(\int_0^1 \sqrt{m(t)} \varphi \left(\sqrt{m(t)} + \frac{y-x}{2\sqrt{m(t)}} \right) dt \right) e^{-x} H(x, y)
\end{aligned}$$

as $n \rightarrow \infty$.

The remainder is to show that (4.9) holds for any fixed $x, y \in \mathbb{R}$. Without loss of generality, we assume that $m(t)$ is increasing.

If $x \leq y$, note that $\int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz$ is increasing about t , so we have

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \\
&= \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_y^\infty \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz dt \\
&> \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^n \int_y^\infty \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \\
&= \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_y^\infty \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz dt \\
&< \int_0^{1+\frac{1}{n}} \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= \int_0^1 \int_y^\infty \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + O\left(\frac{1}{n}\right),
\end{aligned}$$

which implies that (4.9) holds for $x \leq y$.

To verify (4.9) holding for $x > y$, we just need to prove that

$$\frac{1}{n} \sum_{i=1}^n \int_x^y \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz - \int_0^1 \int_x^y \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt = O\left(\frac{1}{n}\right), \quad (4.11)$$

which will be proved in return by the following three cases: (i) $y \leq x - 2m(1)$; (ii) $x - 2m(1) < y < x - 2m(0)$, and (iii) $x - 2m(0) \leq y < x$. In fact, the arguments of (i) and (iii) are similar. The rest is to focus on (i) and (ii).

For case (i), i.e. $y \leq x - 2m(1)$, it is known that $y \leq x - 2m(t) \leq x$ for any $t \in [0, 1]$. Hence,

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \\
&= \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz dt \\
&< \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&< \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= \int_0^1 \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \\
&= \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz dt \\
&> \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_y^{x-2m(t)} \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz dt \\
&> \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\
&= \int_0^1 \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + O\left(\frac{1}{n}\right).
\end{aligned} \tag{4.13}$$

Similarly,

$$\frac{1}{n} \sum_{i=1}^n \int_{x-2m(\frac{i}{n})}^x \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz \leq \int_0^1 \int_{x-2m(t)}^x \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + O\left(\frac{1}{n}\right) \tag{4.14}$$

and

$$\frac{1}{n} \sum_{i=1}^n \int_{x-2m(\frac{i}{n})}^x \Phi \left(\sqrt{m\left(\frac{i}{n}\right)} + \frac{x-z}{2\sqrt{m\left(\frac{i}{n}\right)}} \right) e^{-z} dz > \int_0^1 \int_{x-2m(t)}^x \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt. \tag{4.15}$$

Combining with (4.12)-(4.15), it shows that (4.11) holds for case (i).

Next we consider case (ii), i.e. $x - 2m(1) < y < x - 2m(0)$. Note that there exists $x^* \in (0, 1)$ such that $y = x - 2m(x^*)$ since $m(t)$ is increasing and continuous. Split the following integral into two parts:

$$\begin{aligned} & \int_0^1 \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt \\ &= \int_0^{x^*} \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + \int_{x^*}^1 \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt. \end{aligned}$$

By arguments similar with (4.12)-(4.15), we can get

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^{[nx^*]} \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m(\frac{i}{n})} + \frac{x-z}{2\sqrt{m(\frac{i}{n})}} \right) e^{-z} dz \\ &= \int_0^{x^*} \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + O\left(\frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n} \sum_{i=[nx^*]+1}^n \int_y^{x-2m(\frac{i}{n})} \Phi \left(\sqrt{m(\frac{i}{n})} + \frac{x-z}{2\sqrt{m(\frac{i}{n})}} \right) e^{-z} dz \\ &= \int_{x^*}^1 \int_y^{x-2m(t)} \Phi \left(\sqrt{m(t)} + \frac{x-z}{2\sqrt{m(t)}} \right) e^{-z} dz dt + O\left(\frac{1}{n}\right). \end{aligned}$$

Combining above with (4.14), (4.15), we show that (4.11) holds for case (ii).

Now, (4.11) is derived for any fixed $x, y \in \mathbb{R}$, which complete the proof. \square

Proof of Theorem 3. For fixed $x, y \in \mathbb{R}$, if $\max(x, y) < z < 4 \log b_n$ we have

$$\Phi \left(\frac{u_n(\min(x, y)) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) < \frac{\exp \left(-\frac{b_n^2(1-\rho_{ni})}{4} - \frac{\min(x, y)}{1+\rho_{ni}} + \frac{\rho_{ni} z}{1+\rho_{ni}} \right)}{\sqrt{2\pi} \left(\frac{z - \min(x, y)}{b_n \sqrt{1-\rho_{ni}^2}} - \sqrt{\frac{1-\rho_{ni}}{1+\rho_{ni}}} b_n - \sqrt{\frac{1-\rho_{ni}}{1+\rho_{ni}}} \frac{z}{b_n} \right)}$$

for large n by using Mills' inequality. Combining with (1.4), (4.1) and $\lim_{n \rightarrow \infty} (\log n)^4 \max_{1 \leq i \leq n} m(i/n) = 0$, we have

$$\begin{aligned} & \int_{\max(x, y)}^{4 \log b_n} \Phi \left(\frac{u_n(\min(x, y)) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz \\ &< \frac{(1 + \rho_{ni}) \exp \left(-\frac{b_n^2(1-\rho_{ni})}{4} - \frac{\min(x, y) + \max(x, y)}{1+\rho_{ni}} \right)}{\sqrt{2\pi} \left(\frac{\max(x, y) - \min(x, y)}{b_n \sqrt{1-\rho_{ni}^2}} - \sqrt{\frac{1-\rho_{ni}}{1+\rho_{ni}}} b_n - \sqrt{\frac{1-\rho_{ni}}{1+\rho_{ni}}} \frac{4 \log b_n}{b_n} \right)} \\ &< \frac{2\sqrt{2m(\frac{i}{n})} \left(1 - \frac{\log \log n + \log 4\pi}{4 \log n} + o\left(\frac{1}{\log n}\right) \right) \exp \left(-\frac{1}{2} m\left(\frac{i}{n}\right) \left(1 - \frac{\log \log n + \log 4\pi}{2 \log n} + o\left(\frac{1}{\log n}\right) \right) + \frac{|x+y|}{2 - \frac{m(i/n)}{\log n}} \right)}{\sqrt{\pi} \left(\max(x, y) - \min(x, y) - 2m\left(\frac{i}{n}\right) \left(1 - \frac{\log \log n + \log 4\pi}{2 \log n} + o\left(\frac{1}{\log n}\right) \right) - 4 \frac{m(\frac{i}{n})}{\log n} \log b_n \right)} \\ &< 2\sqrt{2 \max_{1 \leq i \leq n} m\left(\frac{i}{n}\right)} \left(1 - \frac{\log \log n + \log 4\pi}{4 \log n} + o\left(\frac{1}{\log n}\right) \right) \\ &\quad \times \frac{\exp \left(-\frac{1}{2} \min_{1 \leq i \leq n} m\left(\frac{i}{n}\right) \left(1 - \frac{\log \log n + \log 4\pi}{2 \log n} + o\left(\frac{1}{\log n}\right) \right) + \frac{|x+y|}{2 - \frac{\max_{1 \leq i \leq n} m(i/n)}{\log n}} \right)}{\sqrt{\pi} \left(\max(x, y) - \min(x, y) - 2m\left(\frac{i}{n}\right) \left(1 - \frac{\log \log n + \log 4\pi}{2 \log n} + o\left(\frac{1}{\log n}\right) \right) - 4 \frac{\max_{1 \leq i \leq n} m(i/n)}{\log n} \log b_n \right)} \end{aligned}$$

$$= O(b_n^{-4})$$

for any $1 \leq i \leq n$.

Noting that

$$\int_{4 \log b_n}^{\infty} \Phi \left(\frac{u_n(\min(x, y)) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz = O(b_n^{-4})$$

for $1 \leq i \leq n$, we have

$$n^{-1} \sum_{i=1}^n \int_{\max(x, y)}^{\infty} \Phi \left(\frac{u_n(\min(x, y)) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz = O(b_n^{-4})$$

for large n . Hence combining above with (4.5), we can get

$$\begin{aligned} & b_n^2 \left(-\sum_{i=1}^n (1 - F_i(u_n(x), u_n(y))) + e^{-\min(x, y)} \right) \\ &= b_n^2 \left(-n(1 - \Phi(\min(x, y))) + e^{-\min(x, y)} \right) \\ & \quad - b_n^2 n^{-1} (1 - b_n^{-2} + O(b_n^{-4}))^{-1} \sum_{i=1}^n \int_{\max(x, y)}^{\infty} \Phi \left(\frac{u_n(\min(x, y)) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz \\ & \rightarrow \frac{(\min(x, y))^2 + 2 \min(x, y)}{2} e^{-\min(x, y)} \end{aligned}$$

as $n \rightarrow \infty$, which implies (2.2). The proof is complete. \square

Proof of Theorem 4. By Mills' inequality we have

$$1 - \Phi \left(\frac{u_n(y) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) < \frac{\exp \left(-\frac{b_n^2(1 - \rho_{ni})}{2(1 + \rho_{ni})} - \frac{y - \rho_{ni} z}{1 + \rho_{ni}} - \frac{1}{2} \log b_n^2(1 - \rho_{ni}) \right)}{\sqrt{\pi} \left(1 + \frac{y - z}{b_n^2(1 - \rho_{ni})} + \frac{z}{b_n^2} \right)}$$

for large n , which implies that

$$\int_x^{4 \log b_n} \left(1 - \Phi \left(\frac{u_n(y) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz = O(b_n^{-4})$$

for any $1 \leq i \leq n$ due to (1.4), (4.1), $\lim_{n \rightarrow \infty} \min_{1 \leq i \leq n} m(i/n) = \infty$ and $\lim_{n \rightarrow \infty} \frac{\log \log n}{\min_{1 \leq i \leq n} m(i/n)} = 0$.

Combining with

$$\int_{4 \log b_n}^{\infty} \left(1 - \Phi \left(\frac{u_n(y) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz = O(b_n^{-4}),$$

we have

$$\begin{aligned} & \sum_{i=1}^n \mathbb{P}(X_{ni} > u_n(x), Y_{ni} > u_n(y)) \\ &= n^{-1} (1 - b_n^{-2} + O(b_n^{-4}))^{-1} \sum_{i=1}^n \int_x^{\infty} \left(1 - \Phi \left(\frac{u_n(y) - \rho_{ni} u_n(z)}{\sqrt{1 - \rho_{ni}^2}} \right) \right) e^{-z} \exp \left(-\frac{z^2}{2b_n^2} \right) dz \\ &= O(b_n^{-4}). \end{aligned} \tag{4.16}$$

It follows from (4.5) and (4.16) that

$$\begin{aligned}
& b_n^2 \left(-\sum_{i=1}^n (1 - F_i(u_n(x), u_n(y))) + e^{-x} + e^{-y} \right) \\
&= b_n^2 \left(-n(1 - \Phi(u_n(x))) + e^{-x} - n(1 - \Phi(u_n(y))) + e^{-y} + \sum_{i=1}^n \mathbb{P}(X_{ni} > u_n(x), Y_{ni} > u_n(y)) \right) \\
&\rightarrow \frac{x^2 + 2x}{2} e^{-x} + \frac{y^2 + 2y}{2} e^{-y}
\end{aligned}$$

as $n \rightarrow \infty$. Hence (2.3) can be derived, which complete the proof. \square

Proof of Theorem 5. Define

$$\begin{aligned}
Z_i &= -\frac{\rho_{ni}}{(\log n)(1 - \rho_{ni}^2)^2} (X_{ni}^2 + Y_{ni}^2) + \frac{1 + \rho_{ni}^2}{(\log n)(1 - \rho_{ni}^2)^2} X_{ni} Y_{ni} + \frac{\rho_{ni}}{(\log n)(1 - \rho_{ni}^2)} \\
&:= Z_{i,1} + Z_{i,2} + Z_{i,3},
\end{aligned}$$

one can check that

$$\begin{aligned}
\mathbf{E} Z_{i,1}^2 &= \frac{4(\rho_{ni}^4 + 2\rho_{ni}^2)}{(\log n)^2(1 - \rho_{ni}^2)^4}, & \mathbf{E} Z_{i,2}^2 &= \frac{(1 + \rho_{ni}^2)^2}{(\log n)^2(1 - \rho_{ni}^2)^4} (1 + 2\rho_{ni}^2) \\
\mathbf{E} Z_{i,3}^2 &= \frac{\rho_{ni}^2}{(\log n)^2(1 - \rho_{ni}^2)^2}, & \mathbf{E} Z_{i,1} Z_{i,2} &= -\frac{6\rho_{ni}^2(1 + \rho_{ni}^2)}{(\log n)^2(1 - \rho_{ni}^2)^4} \\
\mathbf{E} Z_{i,1} Z_{i,3} &= -\frac{2\rho_{ni}^2}{(\log n)^2(1 - \rho_{ni}^2)^3}, & \mathbf{E} Z_{i,2} Z_{i,3} &= \frac{\rho_{ni}^2(1 + \rho_{ni}^2)}{(\log n)^2(1 - \rho_{ni}^2)^3},
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \mathbf{E} Z_i^2 &= \frac{1}{n} \sum_{i=1}^n \left(\frac{1 - \frac{\alpha + \beta(\frac{i}{n})^\gamma}{\log n} + \frac{(\alpha + \beta(\frac{i}{n})^\gamma)^2}{2(\log n)^2}}{2(\alpha + \beta(\frac{i}{n})^\gamma)^2 \left(1 - \frac{\alpha + \beta(\frac{i}{n})^\gamma}{2 \log n}\right)^2} \right) \\
&\rightarrow \int_0^1 \frac{1}{2(\alpha + \beta t^\gamma)^2} dt
\end{aligned} \tag{4.17}$$

as $n \rightarrow \infty$. It is easy to check that

$$\mathbf{E} \left(\frac{1}{n} \sum_{i=1}^n (Z_i^2 - \mathbf{E} Z_i^2) \right)^2 = \frac{1}{n^2} \sum_{i=1}^n \mathbf{E} (Z_i^2 - \mathbf{E} Z_i^2)^2 = O\left(\frac{1}{n}\right),$$

which combining with (4.17) implies that

$$\sum_{i=1}^n \left(\frac{1}{\sqrt{n}} Z_i \right)^2 \xrightarrow{P} \int_0^1 \frac{1}{2(\alpha + \beta t^\gamma)^2} dt \tag{4.18}$$

Obviously, we have

$$\max_{1 \leq i \leq n} \left| \frac{1}{\sqrt{n}} Z_i \right| \xrightarrow{P} 0 \quad \text{and} \quad \mathbf{E} \left(\max_{1 \leq i \leq n} \frac{1}{n} Z_i^2 \right) = o(1). \tag{4.19}$$

Hence combining with (4.18), (4.19) we can get

$$\frac{1}{\sqrt{n}} l_{n1}(\alpha, \beta, \gamma) \xrightarrow{P} N \left(0, \int_0^1 \frac{1}{2(\alpha + \beta t^\gamma)^2} dt \right) \tag{4.20}$$

as $n \rightarrow \infty$.

Define

$$\begin{aligned} Z_i^* &= -\frac{\left(\frac{i}{n}\right)^\gamma \rho_{ni}}{(\log n)(1-\rho_{ni}^2)^2} (X_{ni}^2 + Y_{ni}^2) + \frac{\left(\frac{i}{n}\right)^\gamma (1+\rho_{ni}^2)}{(\log n)(1-\rho_{ni}^2)^2} X_{ni} Y_{ni} + \frac{\left(\frac{i}{n}\right)^\gamma \rho_{ni}}{(\log n)(1-\rho_{ni}^2)} \\ &:= Z_{i,1}^* + Z_{i,2}^* + Z_{i,3}^*, \end{aligned}$$

and

$$\begin{aligned} Z_i^{**} &= -\frac{\left(\frac{i}{n}\right)^\gamma (\log \frac{i}{n}) \rho_{ni}}{(\log n)(1-\rho_{ni}^2)^2} (X_{ni}^2 + Y_{ni}^2) + \frac{\left(\frac{i}{n}\right)^\gamma (\log \frac{i}{n}) (1+\rho_{ni}^2)}{(\log n)(1-\rho_{ni}^2)^2} X_{ni} Y_{ni} + \frac{\left(\frac{i}{n}\right)^\gamma (\log \frac{i}{n}) \rho_{ni}}{(\log n)(1-\rho_{ni}^2)} \\ &:= Z_{i,1}^{**} + Z_{i,2}^{**} + Z_{i,3}^{**}. \end{aligned}$$

Similar to the proofs of (4.20), we can show that

$$\begin{cases} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} l_{n2}(\alpha, \beta, \gamma) & \stackrel{p}{=} N\left(0, \int_0^1 \frac{t^{2\gamma}}{2(\alpha + \beta t^\gamma)^2} dt\right), \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} l_{n3}(\alpha, \beta, \gamma) & \stackrel{p}{=} N\left(0, \int_0^1 \frac{t^{2\gamma} (\log t)^2}{2(\alpha + \beta t^\gamma)^2} dt\right). \end{cases} \quad (4.21)$$

By arguments similar to (4.17), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{E} Z_i Z_i^* &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^\gamma \mathbf{E} Z_i^2 = \int_0^1 \frac{t^\gamma}{2(\alpha + \beta t^\gamma)^2} dt, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{E} Z_i^* Z_i^{**} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^{2\gamma} \left(\log \frac{i}{n}\right) \mathbf{E} Z_i^2 = \int_0^1 \frac{t^{2\gamma} \log t}{2(\alpha + \beta t^\gamma)^2} dt \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{E} Z_i Z_i^{**} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^\gamma \left(\log \frac{i}{n}\right) \mathbf{E} Z_i^2 = \int_0^1 \frac{t^\gamma \log t}{2(\alpha + \beta t^\gamma)^2} dt.$$

Hence, by Cramér device, we can derive that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} (l_{n1}(\alpha, \beta, \gamma), l_{n2}(\alpha, \beta, \gamma), l_{n3}(\alpha, \beta, \gamma))^T \stackrel{d}{=} N(0, \Sigma), \quad (4.22)$$

where Σ is given by (2.6).

It is straight forward to check that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \begin{pmatrix} \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \alpha} & \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \beta} & \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \gamma} \\ \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \alpha} & \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \beta} & \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \gamma} \\ \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \alpha} & \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \beta} & \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \gamma} \end{pmatrix} \\ & \stackrel{p}{=} \begin{pmatrix} \int_0^1 \frac{1}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^\gamma}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{\beta t^\gamma \log t}{2(\alpha + \beta t^\gamma)^2} dt \\ \int_0^1 \frac{t^\gamma}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma}}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{\beta t^{2\gamma} \log t}{2(\alpha + \beta t^\gamma)^2} dt \\ \int_0^1 \frac{t^\gamma \log t}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{t^{2\gamma} \log t}{2(\alpha + \beta t^\gamma)^2} dt & \int_0^1 \frac{\beta t^{2\gamma} (\log t)^2}{2(\alpha + \beta t^\gamma)^2} dt \end{pmatrix} \\ & := \Delta. \end{aligned} \quad (4.23)$$

Hence, the desired result is derived by (4.22), (4.23) and Taylor expansion. The proof is complete. \square

Proof of Theorem 6. It follows from the proof of Theorem 5 with known $\gamma = 1$. \square

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